



B.Sc. EXAMINATION

MAS 204 Calculus III

Thursday, 11 May 2006 10:00-13:00

Duration: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

SECTION A

You should attempt all questions. Marks awarded are shown next to the questions.

1. [7 marks] Using index notation show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

You may assume standard index identity: $\varepsilon_{ikn}\varepsilon_{ilm} = \delta_{kl}\delta_{nm} - \delta_{km}\delta_{nl}$.

2. [10 marks] Consider the sphere $x^2 + y^2 + z^2 + \frac{3}{2}y - 2z = 0$.
- (a) Write down an equation for a plane tangent to the sphere at the point (1,0,1).
 - (b) Write down an equation for the line which is normal to this plane and passes through the center of the sphere.
 - (c) Show that this line passes through the point (1,0,1).
3. [6 marks] Two different vector fields are given:
 $\mathbf{F}_1 = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{F}_2 = y\mathbf{i} - x\mathbf{j}$.
- (a) Which vector field has zero/nonzero divergence.
 - (b) Which vector field has zero/nonzero curl.
4. [8 marks] Using integral theorems, simplify the following:
- (a) $\oint (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot d\mathbf{S}$ over an arbitrary closed surface. Express your result in terms of the volume within this closed surface.
 - (b) $\oint (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot d\mathbf{r}$ over an arbitrary closed path.
5. [8 marks] Without taking any integrals express the following as Fourier series:
- (a) $\cos(2x + \frac{\pi}{3})$, (b) $\cos^3 x$.
6. [9 marks] Consider a line integral $\int \mathbf{F} \cdot d\mathbf{r}$ of each of the following vector fields:
- (a) $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$,
 - (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + xy\mathbf{k}$,
 - (c) $\mathbf{F} = \frac{1}{2}y^2z^2\mathbf{i} + xyz^2\mathbf{j} + xy^2z\mathbf{k}$.

In each case, state whether the line integral will depend on the path and give brief reasons in your answer.

7. [12 marks] We want to find a path $y(x)$ between two given endpoints such that

$$\int [(y')^2 - 2y^2] dx$$

is extremized. [Here $y' = \frac{dy}{dx}$ as usual.] Explain that to solve the problem one can use the Hamiltonian $H = y' \frac{\partial L}{\partial y'} - L = \text{const}$ instead of the Lagrangian L . Find the general solution and then the particular solution if the endpoints are $(0, 0)$ and $(\frac{\pi}{2\sqrt{2}}, 1)$.

SECTION B

Each question carries 20 marks. You may attempt all questions, but only marks for the best TWO questions will be counted.

1. Show that

$$f(x) = \begin{cases} (\pi + x)^2 & \text{for } -\pi \leq x \leq 0 \\ \pi^2 - x^2 & \text{for } 0 < x \leq \pi \end{cases}$$

has Fourier expansion

$$4 \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} \left(\cos nx + \frac{2}{\pi n} \sin nx \right).$$

2. Consider the two-dimensional coordinate system (u, v) defined through

$$x = \cos u \cos v, \quad y = \cos u \sin v.$$

Show that this coordinate system is orthogonal.

3. Using index notations show that

$$(a) \nabla \cdot \nabla \times \mathbf{F} = 0,$$

$$(b) \nabla \times \nabla \Phi = 0,$$

where $\mathbf{F}$ is an arbitrary vector field and Φ is an arbitrary scalar field.

4. The function $\Phi(x, y) = a(x^n - y^m) + bx^ny^m + c\frac{x^n}{y^m}$, (where a , b and c are arbitrary constants and $0 \leq n$, $0 \leq m$ are constant integers) satisfies Laplace's equation in Cartesian coordinates. Find all possible n and m , if (a) $a \neq 0$, $b = c = 0$; (b) $b \neq 0$, $a = c = 0$; (c) $c \neq 0$, $a = b = 0$;

5. The function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ and has boundary conditions

$$\begin{cases} F(x, 1) = x \\ F(0, y) = 0 \\ F(x, 0) = 0 \\ F(1, y) = y \end{cases}$$

Show that

$$F(x, y) = xy.$$

6. A body moves from the point $(1, 1)$ to the point $(3, 2)$. Assuming that the speed of the body at any point (x, y) is equal to $\frac{1}{\sqrt{y}}$ show that the particular curve which extremizes the travel time is the parabola

$$y = 1 + \left(\frac{x-1}{2}\right)^2.$$